Geometrical orbits in the power spectra of waves

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We investigate the relationship between the power spectrum of a wave field and that of a spatially uncorrelated source exciting it. Temporal correlations between values of the field at different positions are also examined. It is found that interference effects can significantly alter the structure of the power spectrum, leading to oscillations in it, even when the power spectrum of the source is a smooth function of frequency. We derive a semiclassical approximation in which these oscillations are related to orbits of the geometrical limit of the wave system. We also derive a trace formula that approximates a spatial average of the wave power spectrum as a sum over periodic orbits. These calculations explain the structure of a measured power spectrum of the fluctuating height of a fluid surface, generated by the circular hydraulic jump, which provided the motivation for the study. [S1063-651X(97)04505-4]

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I. INTRODUCTION

Semiclassical approximations to quantum mechanics have excited a lot of activity in recent years, especially in relation to systems for which the classical dynamics is chaotic [1]. While it is natural for historical reasons to apply semiclassical techniques to quantum mechanics, there is no reason why they cannot be used with equal enthusiasm for other wave systems, such as electromagnetic cavities or acoustic excitation of solid objects. Indeed, even though theoretical studies have tended to concentrate on quantum mechanics, much of the recent experimental work in the field has been with these other wave systems [2–6], so classical wave systems would seem to provide a rich arena for experimental applications of the techniques. There has been a tendency to view these experiments as analogs to quantum mechanical systems and therefore to perform measurements that are natural in that context. The semiclassical approximation of diverse classical wave systems is interesting in its own right, however, and when viewed independently, alternative measurements may be natural that have no analog in quantum mechanics.

An example of such a measurement is one that forms the basis for the calculations in this paper — the characterization of a stochastic variable by its power spectrum. While power spectra are sometimes used in quantum mechanics to extract information about eigenstates, they are used as a tool in the analysis of stochastic systems with a fundamentally different underlying philosophy, the interest in the latter case often focusing on broad scaling properties rather than in identifying peaks at precise frequencies. If the stochastic variable examined has a wave character, interference effects can have a dramatic impact. In this paper we study fluctuations as a function of frequency that are observed in such power spectra, and relate them to trajectories in the geometrical limit of the wave system.

The analysis presented here was inspired by the observation of such effects in a particular experiment [7], described in Sec. II. There a time series was measured of the height of flowing water outside a circular hydraulic jump, from which a power spectrum was computed. This revealed a strong, regular oscillation in frequency, whose period depended on the point where the measurement was taken. It was found that all the essential features of this power spectrum could be explained with a simple model — that of a linearized wave equation describing surface oscillations, driven by a noisy source term modeling the effect of the forcing of the surface oscillations by turbulence generated at the jump. Semiclassical analysis relates the power spectrum to ''classical'' orbits in the geometrical limit, and the oscillation with frequency can be explained with a single orbit deduced from the geometry of the experiment.

This analysis depends little on the details of the particular system, and is developed quite generally. We mention in Sec. II the possibility of observing these effects in blackbody radiation, and in acoustic emissions in solids, for example. We assume a wave system driven by a stochastic source. In Sec. III the power spectrum of the wave field is related to that of the source through an integral of the Green's function over its source variable, under the assumption that the source is uncorrelated at length scales of the order of a wavelength. There is a similar calculation for correlation functions. Through this integral over the Green's function, interference effects appear in the wave field's power spectrum. These are best understood through an approximation for the Green's function as a sum over geometrical orbits. We show in Sec. IV that fluctuations in the power spectrum at a given point can be expressed as a sum over geometrical orbits that begin and end at that point, and whose past passes through the region of excitation of the wave field. These fluctuations are superimposed on a smooth contribution that is reminiscent of the Thomas-Fermi density of states in quantum mechanics. We also show that correlations between two different points can be approximated by a sum over trajectories that connect those two points. Correlations between two different points have no smooth background. We estimate the expressions for the power spectrum in the particular case of billiard systems — homogeneous and isotropic wave equations with interfer-

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ence effects arising from reflections at surfaces where boundary conditions are imposed. The smooth part represents modification of the source power spectrum by a factor that is an algebraic function of frequency, and oscillations are found to have amplitudes that decay exponentially with frequency due to dissipative damping of the wave. If the wave field is measured outside of the region of excitation, however, then even the smooth part has an exponential dependence on frequency.

More elegant semiclassical formulas are obtained by averaging the power spectrum over space. Then the approximation turns into a sum over orbits that are periodic in phase space, and this is calculated in Sec. V. The formula derived there is very much like the Gutzwiller trace formula [1], except that each periodic orbit term carries an extra weight determined by the strength of the source power spectrum over its length. Finally, in Sec. VI we apply the results derived here to understanding quantitatively the main features of the power spectrum measured in the hydraulic jump experiment. We relegate many details of the semiclassical calculation to the Appendixes. In Appendix A, we give a detailed description of the semiclassical calculation for the Green's function — only the main points are discussed in the text. The contributions to the Green's function are damped due to dissipation and in Appendix B we give a simple recipe for calculating the damping factors without having to calculate trajectories in complex phase space. In Appendix C we use the detailed specification of the Green's function discussed in Appendix A to simplify the calculation of the smooth part of the power spectrum. We do the same thing for the oscillating part in Appendix D.

II. MOTIVATION

We begin with a brief discussion of the experimental measurements that provoked the analysis presented in this paper — a complete description of the experiment, with more detailed analysis, can be found in Ref. [7].

In the experiment, measurements were taken of the fluid height at a point outside a circular hydraulic jump. This jump appears when fluid flows radially outwards from a point of injection at the center of a circular plate. At a certain radius, depending on the flux, the fluid height increases abruptly and the radial velocity falls. The flow appears to be stationary at sufficiently small flow rates, but when the flux is large enough the jump starts to fluctuate and waves appear on the surface. This onset of irregular behavior is very interesting as a simple and controllable example of a hydrodynamic instability, and was studied experimentally for this reason. A natural first step in the analysis is to compute a power spectrum from a time series of the height of the fluid at a given point.

When this was done, the power spectrum was seen to exhibit a marked and regular oscillation with frequency. We show in Fig. 1 an example in which the boundary conditions are fixed by placing a wall at the edge of the plate. It is found that the strength of the oscillation effects depends on the manner in which surface waves are reflected from the edge, and placing a wall there is the cleanest way to control the reflections. In the original experiment, reflections from the edge of the plate produced similar, but less controllable effects. In general one finds that the spacing between peaks increases steadily with frequency, roughly as $\Delta\omega \sim \omega^{1/3}$, and it is further observed that the spacing increases when the point of measurement is brought closer to the edge of the plate. This structure can completely dominate the power spectrum, and a full understanding is necessary before an analysis of the underlying phenomena can be attempted.

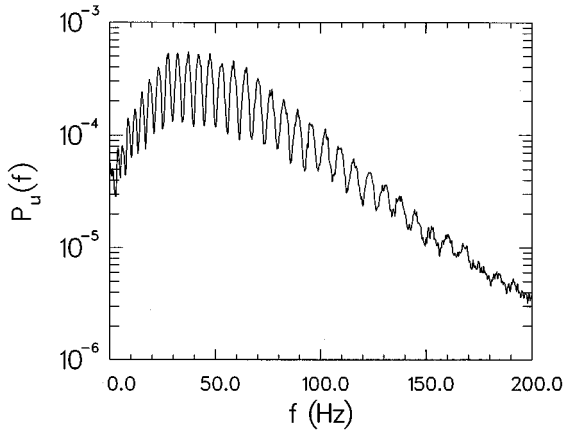

FIG. 1. A power spectrum from the hydraulic jump experiment. In this case the boundary conditions were fixed by placing a reflector near the edge of the plate.

We can explain the main features of the power spectrum with a simple model. We assume that the surface waves are of small enough amplitude that their propagation is governed by a linear wave equation. We then model the driving of the surface waves by the irregular jump by including a noisy source term in this wave equation. From here it is possible to see that the oscillations arise from interference between waves that travel directly from the jump to the point of measurement, and waves that pass throught the point of measurement, are reflected, and return. We present a discussion in the next section that is valid for any linear wave equation driven by noise. Following the general analysis, we will return to the experiment in Sec. VI and show how the model quantitatively explains the main features observed.

Among other possible instances where structures like this might appear, perhaps the most obvious is in the thermal fluctuations of the electromagnetic field of a blackbody cavity. The natural frequency range in which to observe this effect is that of microwaves. In this case, the source exciting the wave would be currents on the surface of the cavity. One might then see in the power spectrum a smooth background coming from the Planck distribution, with interference-induced fluctuations superimposed upon it. Another possibility is the phenomenon of acoustic emission, where the stochastic excitation of sound waves in solids arises through the random changes of domain walls [8]. If there are regimes where the domain walls are of small enough length scale, that source is spatially uncorrelated at length scales of an acoustic wavelength (as demanded by the coming analysis), and if the jumps are close enough together in time, then the effect will be of acoustic noise with a broadband spectrum. Other examples where interference effects might be observed are water waves on longer lengths scales than are present in the hydraulic jump experiment, such as wind-generated ocean waves, seismic waves produced by earthquakes or

other sources, and production of electromagnetic radiation by the currents generated in a turbulent plasma.

III. A GENERAL MODEL

In this section we examine a simple model inspired by the hydraulic jump experiment — that of a field $u(x,t)$ whose evolution in time is governed by a linear wave equation driven by noise. We will concentrate on the following formulation of the problem:

$$\mathcal{L}\left(\frac{\partial}{\partial t},\nabla,x\right)u(x,t)=\eta(x,t), \tag{1}$$

where the linear wave operator $\mathcal{L}(\partial/\partial t,\nabla,x)$ is homogeneous in time and of order n, defined on a d-dimensional configuration space. The source term $\eta(x,t)$ is taken from some ensemble of random functions whose properties will be discussed in more detail later. We will specify more precise conditions on the wave equation in the next section, but for the moment we will just say that it should have solutions that are predominantly oscillatory, with some small degree of dissipation present.

The idea here is that $\eta(x,t)$ might represent some physical process where activity occurs on all possible time scales, possibly with some interesting scaling behavior. Let us suppose that one was interested in measuring the power spectrum of this process, and that it was accessible not directly but through the wave field $u(x,t)$ excited by it. The question then is whether the power spectrum of $u(x,t)$ differs significantly from that of $\eta(x,t)$. We will find that the answer is, in many cases, yes. Because $u(x,t)$ has a wave character, we will find that interference effects can significantly alter the power spectrum. If, for example, the power spectrum of $\eta(x,t)$ decays smoothly to 0 (as $\omega^{-\alpha}$, for example) not only will the power spectrum of $u(x,t)$ decay with a somewhat different rate on average, but significant fluctuations will be superimposed.

It should be pointed out that in many physical systems of interest, the wave equation will not be in the form of a partial differential equation as in Eq. (1) — it might be an integral equation or, as in the case of surface waves in a liquid, have undetermined boundary conditions as an integral part of the time evolution. Even if in the form of a partial differential equation (PDE), the wave equation might not, as implied by the notation used so far, be a scalar wave equation — sound waves in solids or electromagnetic waves are governed by vector wave equations, for example. We assume a scalar partial differential equation because we can then, in the semiclassical analysis of the next section, carry over standard formulas from the WKB approximation in quantum mechanics with minimal modification or awkwardness of notation. When more general wave systems are to be considered, the basic structure of the results will be the same.

A. Calculation of the power spectrum

Let us first relate the power spectrum of $u(x,t)$ to that of $\eta(x,t)$. The autocorrelation function of the source is assumed to be of the form

$$\langle\,\eta(x,t)\,\eta(x',t')\rangle=C_\eta(x,x',t-t') \tag{2}$$

and the Fourier transform of this gives

$$\langle\,\tilde{\eta}(x,\omega)\,\tilde{\eta}^*(x',\omega')\rangle=2\pi\tilde{C}_\eta(x,x',\omega)\,\delta(\omega-\omega'). \tag{3}$$

There are similar expressions for the autocorrelation function of the response $u(x,t)$.

To connect the autocorrelation functions we write the wave equation in the frequency domain:

$$\mathcal{L}(i\omega,\nabla,x)\tilde{u}(x,\omega)=\tilde{\eta}(x,\omega), \tag{4}$$

and define its Green's function $G(x,x',\omega)$ in the usual manner,

$$\mathcal{L}(i\omega,\nabla,x)G(x,x',\omega)=\delta(x-x'). \tag{5}$$

As an abstract operator, we will sometimes denote the wave operator at the left by $\hat{\mathcal{L}}(\omega)$. The Green's function is formed by taking matrix elements of the related operator $\hat{G}(\omega)=\hat{\mathcal{L}}(\omega)^{-1}$.

Using the Green's function to solve for the response and averaging over the ensemble of source functions, we get the following relationship between autocorrelation functions:

$$\tilde{C}_u(x_1,x_2,\omega)=\int dx_1'\int dx_2'\;G(x_1,x_1',\omega)G^*(x_2,x_2',\omega)\times\tilde{C}_\eta(x_1',x_2',\omega). \tag{6}$$

We will now commit ourselves to the assumption that the spatial correlation length of the source is much smaller than typical wavelengths observed in the wave system. This allows us to make the replacement

$$\tilde{C}_\eta(x_1',x_2',\omega)\to P_\eta(x_1',\omega)\,\delta(x_1'-x_2') \tag{7}$$

in Eq. (6), where we will refer to $P_\eta(x,\omega)$ as the power spectrum of the source. Then,

$$\tilde{C}_u(x_1,x_2,\omega)=\int dx'\,G(x_1,x',\omega)G^*(x_2,x',\omega)P_\eta(x',\omega). \tag{8}$$

The most important case is when $x_1=x_2$ and this gives us an expression for $P_u(x,\omega)=\tilde{C}_u(x,x,\omega)$, which we will refer to as the power spectrum of the wave system:

$$P_u(x,\omega)=\int dx'\,|G(x,x',\omega)|^2P_\eta(x',\omega). \tag{9}$$

Notice that the normalizations of the source and wave power spectra are defined differently. These expressions for the autocorrelation function and power spectrum will form the basis for the semiclassical approximation in the following section. Through them, interference effects in the Green's function become manifest in the power spectrum of the wave.

To accentuate the structure of the basic equations, let us consider for the moment the case where $P_\eta(x,\omega)=P_\eta(\omega)$ is position independent. Then Eq. (8) can be written simply in bra-ket notation as follows:

$$\tilde{C}_u(x_1,x_2,\omega)=P_\eta(\omega)\langle x_1|\hat{G}(\omega)\hat{G}^\dagger(\omega)|x_2\rangle. \tag{10}$$

So in this case the modulation factor for $\widetilde{C}(x_1,x_2,\omega)$ is just a matrix element of the operator $\hat{G}(\omega)\hat{G}^{\dagger}(\omega)$ and the particular case of the power spectrum corresponds to diagonal matrix elements of this operator. In the general case of position-dependent $P_{\eta}(x,\omega)$, one has to sandwich another operator between $\hat{G}(\omega)$ and $\hat{G}^{\dagger}(\omega)$ to represent the modification of the measure dx' by the factor $P_{\eta}(x',\omega)$.

IV. SEMICLASSICAL ANALYSIS

In quantum mechanics, there is a well understood approximation to the Green's function in terms of trajectories of the corresponding classical system [1]. We will use the analogous approximation for $G(x,x',\omega)$ to interpret the interference effects arising in the power spectrum of $u(x,t)$. The most important features can be understood without reference to many of the technical details arising in the calculation. We will therefore present the calculation along two tracks. In the main text we will give the basic structure of the approximations, writing down WKB approximations, not always specifying in detail amplitudes and phase shifts, but presenting enough detail that the basic qualitative structure is clear. We also give a detailed account of the manipulation of amplitudes and phase shifts, but this is relegated to the Appendixes as the calculation proceeds.

A. The starting approximation

Approximation of the Green's function in terms of orbits can be achieved by direct comparison with the analogous quantum mechanical calculation. This is described in detail in Appendix A. We work under the assumption that the length scales over which the wave medium changes are much larger than typical wavelengths. Under these conditions, an eikonal ansatz $A(x)\exp[iS(x)]$ is justified as a local approximation to the wave field. Insertion of the eikonal ansatz into the wave equation will yield a local dispersion relation $\omega=\nu(x,k)$, where $k=\nabla S$ is the local wave vector. The function $\nu(x,k)$ will then serve as a Hamiltonian for the ''classical limit'' of the wave equation — generating trajectories or ''rays'' $[x(t),k(t)]$ in the $2d$-dimensional phase space for which (x,k) are canonical coordinates. Often there will be no explicit position dependence in the dispersion relation but interesting dynamics will occur at surfaces where boundary conditions are imposed and waves are reflected, as in the billiard problems of quantum mechanics.

We will assume that the operator $\hat{\mathcal{L}}(\omega)$ is approximately Hermitean — so the wave equation will have truly wavelike solutions. A small non-Hermitean part should be present to represent the effects of dissipation, but we will always treat this perturbatively. If there is a small anti-Hermitean component to $\hat{\mathcal{L}}(\omega)$, then the dispersion relation will have a small imaginary part, leading in turn to rays $[x(t),k(t)]$ that are slightly complex.

The approximation for the Green's function, valid when the dispersion relation changes over spatial scales much larger than a wavelength, is of the form

$$G(x,x',\omega)\approx\sum_{\alpha} A_{\alpha}(x,x',\omega)e^{iS_{\alpha}(x,x',\omega)}, \tag{11}$$

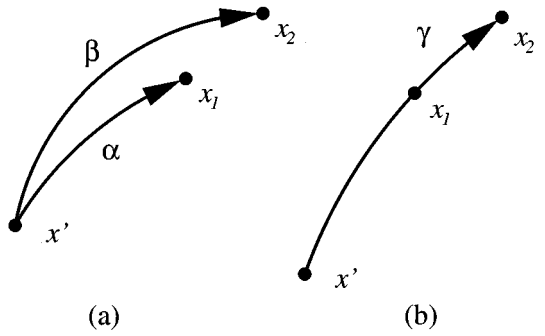

FIG. 2. Before stationary phase integration of Eq. (13), pairs of distinct orbits α and β contribute to the correlation function, as illustrated schematically in (a). A contribution with stationary phase is obtained when the initial conditions of these orbits coincide, giving a sum over the histories of single orbits γ shown in (b).

where α labels orbits going from x' to x at the specified value of $\omega=\nu(x,k)$, and $S_{\alpha}(x,x',\omega)=\int_{x'}^{x}k\,dx$, taken along the orbit. In this preliminary analysis we will suppress Maslov indices (by incorporating them into the amplitude) and not specify $A_{\alpha}(x,x',\omega)$ any further other than to say that it is a function that changes over spatial scales much longer than a wavelength — a complete description is given in Appendix A. In a physical problem, dissipation will be present and the trajectories will have small imaginary components. We denote the real and imaginary parts of the action by

$$S=R+iK, \tag{12}$$

with K small and positive. The oscillatory part is the most important and can be treated at lowest order by ignoring the anti-Hermitean part of the wave operator and getting real trajectories from the corresponding real dispersion relation. Then the decay can be added in by calculating the imaginary part of the action as a perturbation of this real orbit — an explicit formula is given in Appendix B. This decay is in turn important for the convergence of the integrations to follow.

We are now ready to start calculating the correlation function. Under the approximation above, Eq. (8) becomes

$$\widetilde{C}(x_1,x_2,\omega)=\sum_{\alpha\beta}\int dx' A_{\alpha}(x_1,x',\omega)A^{*}_{\beta}(x_2,x',\omega) \times P_{\eta}(x',\omega)e^{iS_{\alpha}(x_1,x',\omega)-iS^{*}_{\beta}(x_2,x',\omega)}, \tag{13}$$

where α and β label the trajectories from x' to x_1 and x_2, respectively, illustrated schematically in Fig. 2(a). Under typical semiclassical conditions, this integral will be dominated by the contributions from regions where the phase is stationary. In principle, significant contributions can also arise from boundaries of the integration where the geometry of the system leads to a cutoff of trajectories [9]. (Such effects might be observed in the hydraulic jump experiment, for example [7].) In this paper we concentrate on the dominant contributions of stationary phase points. We assume that the length scales over which $P_{\eta}(x,\omega)$ varies are long, so that it does not enter into the stationary phase condition. Gener-

ating function conditions exist for the action integrals stating that $\partial R(x,x',\omega)/\partial x = k$ and $\partial R(x,x',\omega)/\partial x' = -k'$, where k' and k are respectively the initial and final wave vectors of the orbit (in real dynamics) [1], and as a result the stationary phase condition for the integral above can be written (ignoring the imaginary part of the action) as

$$0 = \frac{\partial R_\alpha(x_1,x',\omega)}{\partial x'} - \frac{\partial R_\beta(x_2,x',\omega)}{\partial x'} = -k_1'(x_1,x',\omega) + k_2'(x_2,x',\omega). \tag{14}$$

Therefore the initial wave vectors of the orbits should be the same, which simply means that the points x_1 and x_2 lie on different parts of the same trajectory, as illustrated in Fig. 2(b). Thus, to find the stationary phase points, we must first find the trajectories going from x_1 to x_2 or *vice versa* — then the stationary phase points consist of all those positions x' on the ''past'' of that trajectory, preceding both x_1 and x_2.

It is possible to imagine circumstances where $P_\eta(x,\omega)$ would have short length scales such as, for example, being restricted to a boundary (as for blackbody radiation in a cavity, and possibly the hydraulic jump experiment). If the source is restricted to a surface of codimension 1, the analysis is not altered considerably. We apply the stationary phase condition only along directions parallel to this surface, implying that those $d-1$ components of momentum match. Since the trajectories should also have the same frequency, we are still essentially restricted to pairs of trajectories with matching initial conditions. Therefore the orbits contributing to the correlation function are the same as before, but the integral over x' is restricted to the boundary.

We should point out that the products of Green's functions that we encounter here, and the subsequent orbit sums developed in the following calculation, bear a strong formal resemblance to certain quantum mechanical calculations. In a semiclassical analysis of systems with small scattering centers or sharp corners in boundaries, diffraction effects are included by concatenating separate Green's functions and including ''diffraction coefficients'' for each scattering point [9]. When computing traces the resulting integration of products of Green's functions is similar to that presented here. In the analysis of conductance fluctuations of two-dimensional electron gasses products of Green's functions are also encountered [10] leading likewise to simple orbit sums of the type presented here.

B. The smooth contribution

Before proceeding further with the approximation of Eq. (13) for general orbits, we single out a special contribution that arises in the approximation of the power spectrum, when $x_1 = x_2$. Then it is possible for the orbits α and β in Eq. (13) to be exactly the same orbit (with the same length as well as the same initial conditions). In this case the actions cancel and there is no phase variation as x' is varied. Then the integral must be computed as a full d-dimensional integral, giving the following smooth contribution to the power spectrum:

$$P_u^{\mathrm{sm}}(x,\omega) = \sum_\alpha \int dx' |A_\alpha(x,x',\omega)|^2 e^{-2K_\alpha(x,x',\omega)} P_\eta(x',\omega). \tag{15}$$

Superimposed upon this nonoscillating function of x will be oscillating contributions from cross terms $\alpha \neq \beta$, to be calculated later. This contribution plays a role analogous to that of the Thomas-Fermi term in the trace formula for the density of states. There is an important difference however. In the formula above, it is assumed that the integral is dominated by orbits that are longer than a few wavelengths — that the contribution from shorter orbits, for which the semiclassical approximation is not valid and $A_\alpha(x,x',\omega)$ diverges, is negligible. By contrast, it is precisely the short orbit contribution that is responsible for the Thomas-Fermi density of states [1]. Besides the differences in origin, though, the contributions are structurally similar. This will be especially evident when we discuss traces in Sec. V, where the smooth contribution will be shown to consist of an integral over a manifold of fixed frequency in phase space, with the same measure that occurs in the Thomas-Fermi density of states.

In its present form, Eq. (15) is impractical because the amplitude $A_\alpha(x,x',\omega)$ is a complicated function of its arguments. We show in Appendix C, however, that the amplitude is proportional to a Jacobian that transforms the integral over dx' into an integral over phase space coordinates that is much simpler. This new form is

$$P_u^{\mathrm{sm}}(x_0,\omega_0) = \frac{1}{(2\pi)^{d-1}} \int dx'\,dk'\,dt\,\delta(\omega-\omega_0) \times \delta(x(t)-x_0)|JJ'|e^{-2K(x',k',t)} P_\eta(x',\omega), \tag{16}$$

where, for the moment, the symbols x and ω are reserved for variables in the integration, and x_0 and ω_0 are the particular values at which the power spectrum is to be evaluated. The integral is over the $(2d+1)$-dimensional space of all possible trajectories, parametrized by the initial conditions (x',k') and the time t. The δ functions restrict the integral to the submanifold where the frequency of the trajectory is ω_0 and the final position is x_0. The arguments of the imaginary part K of the action function were rewritten to coincide with the integration variables, but it still just comes from an integral of $k\cdot dx$ along the orbit (the subscript α is dropped because there is now a unique trajectory for each argument).

The factors J' and J are Jacobians to be evaluated, respectively, at the initial and final points of the trajectory — they express the relationship $d\tau/dt$ between increments in time t and in another timelike coordinate τ. It can be calculated as follows. The ''classical'' limit of the wave operator $\hat{\mathcal{L}}(\omega)$ yields a dispersion function $D(x,k,\omega)$ and τ is the time coordinate when this is used as a Hamiltonian. At a point (x,k) in phase space, the Jacobian can be calculated from $J = -[\partial D(x,k,\omega)/\partial\omega]^{-1}$. The coordinate τ and dispersion function $D(x,k,\omega)$ are central to the discussion introducing the Green's-function approximation in Appendix A, and the reader should look there for a more thorough discussion.

One can also parametrize the trajectories by the final phase space coordinates (x,k) rather than the initial coordinates (x',k'), and because time evolution in phase space is symplectic, the Jacobian for this transformation is 1. Following this coordinate transformation, the integral over x annihilates the δ function in that variable, giving

$$P_u^{\rm sm}(x,\omega)=\frac{1}{(2\pi)^{d-1}}\int dk\ \delta(\nu(x,k)-\omega)|J| \times\int_{-\infty}^{0}dt'|J'|e^{-2K(x,k,t')}P_\eta(x'(t'),\omega). \tag{17}$$

We have dropped the subscript 0 on the arguments of the power spectrum. We have also redefined the time coordinate to t' so that it varies from $-\infty$ in the infinite past of the trajectory to 0 when the trajectory reaches x. This is now explicitly an integral over all trajectories passing through x with frequency ω — through the measure $\int dk\,\delta(\nu(x,k)-\omega)$ — with each trajectory contributing an integral over its past time coordinate t'.

We can absorb the factor $|J'|$ into the integration measure by writing

$$d\tau'=dt'|J'|. \tag{18}$$

While this has the disadvantage that it obscures the more physical variable t', it is neater, and we will use it from now on. (We will still write time-dependent arguments in terms of t' rather than τ'.) One can also simplify the δ function by writing it in terms of the dispersion function according to $|J|\,\delta(\nu(x,k)-\omega)=\delta(D(x,k,\omega))$. Let us also introduce a special symbol for the integral over τ'. For any phase space point (x,k) we define

$$Q(x,k,\omega)=\int_{-\infty}^{0}d\tau' e^{-2K(x,k,t')}P_\eta(x'(t'),\omega), \tag{19}$$

which is an integral over the past of the trajectory going through (x,k) at $t'=0$. This quantity will turn out to be important also for the oscillating contributions. If the excitation is along a boundary rather than in the bulk, there is an analogous discrete sum over past intersections of the trajectory with the boundary. The smooth contribution can now be written more compactly as

$$P_u^{\rm sm}(x,\omega)=\frac{1}{(2\pi)^{d-1}}\int dk\,\delta(D(x,k,\omega))Q(x,k,\omega). \tag{20}$$

Let us calculate the smooth term in a simple special case. We consider a system that is spatially homogeneous and isotropic, so the dispersion relation $\omega(k)$ is a function of $|k|$ only. We assume that boundary conditions are responsible for any interference effects, so rays are reflected at boundaries but otherwise travel in straight lines, as in quantum billiards. The results below also work for open systems, and give a complete picture in that case because no interference effects are present. We assume that the source is homogeneous, so its power spectrum can be written as $P_\eta(\omega)$. For an orbit of time t, the physical length l is related to it by $l=vt=(\partial\omega/\partial k)t$ and we can write, for the imaginary part of the action,

$$K=\alpha(\omega)l=\alpha(\omega)\frac{\partial\omega}{\partial k}t, \tag{21}$$

where we denote $\alpha(\omega)=\mathrm{Im}[k(\omega)]$. We can now explicitly integrate Eq. (19) for Q, giving the following function of ω alone:

$$Q(\omega)=\frac{P_\eta(\omega)}{\alpha(\omega)U(\omega)}. \tag{22}$$

Here $U(\omega)=|J|^{-1}\partial\omega/\partial k$ is a rescaling of the group velocity corresponding to a variable change $t\to\tau$. The remaining integral over k in Eq. (20) can easily be computed in polar coordinates to give

$$P_u^{\rm sm}(\omega)=\frac{\Omega_d}{(2\pi)^{d-1}}\frac{k(\omega)^{d-1}}{\alpha(\omega)U(\omega)^2}P_\eta(\omega), \tag{23}$$

where Ω_d is the integral over angular coordinates. If we further assume a power-law dispersion relation $\omega\sim k^\alpha$, with $\alpha\sim\omega^\beta$ and $\partial D/\partial\omega\sim\omega^{n-1}$, then

$$P_u^{\rm sm}(\omega)\sim\omega^\delta P_\eta(\omega), \tag{24}$$

with $\delta=(d+1)/\alpha-2n-\beta$. That is, there is an algebraic correction to the power spectrum. This is expected to be typical of cases where the power spectrum of the wave field is measured in the region of excitation. The result would be quite different if the field were excited in one place and measured in another — as is the case in the hydraulic jump experiment. Then $Q(\omega)$ receives a contribution only from the part of a trajectory passing through the excitation region and this would decay exponentially as $e^{-2\alpha(\omega)l}$, where l is the physical distance from the point of measurement to the region of excitation. This exponential modification represents a dramatic change if the source itself has a power spectrum in the form of a power law.

The correlation function $\tilde{C}(x_1,x_2,\omega)$ does not have a smooth background analogous to $P_u^{\rm sm}(x,\omega)$ when there is significant distance — more than a few wavelengths — between x_1 and x_2. This is related to the fact that the analogous correlation function defined for the source itself vanishes at these distances. If, however, we let x_1 approach x_2, then the phase cancellation in the integral over x' between pairs of orbits α and β that are approximately equal becomes weak, and a significant contribution is obtained even though the stationary phase condition is not satisfied. By approximating both amplitudes by that for a trajectory to the mean position $\bar{x}=(x_1+x_2)/2$, one can express the contribution to $\tilde{C}(x_1,x_2,\omega)$ as an integral similar to Eq. (20), but with an additional factor $\exp(i\Delta R)$ for the slight phase variation. An expansion of the phase about this mean point gives $\Delta R=-\bar{k}\cdot\Delta x+O(|\Delta x|^3)$, where $\bar{k}=(k_1+k_2)/2$ and $\Delta x=x_2-x_1$. To get an idea of how this integral behaves, let us assume that the dispersion relation, and also Q, depends on k only through its magnitude k. (It is unlikely that Q would not depend to some extent on the orientation of k, but

the following result should be representative of cases where this orientational dependence is weak.) Then we can write for the smooth part of the correlation function,

$$\widetilde{C}^{\rm sm}(x_1,x_2,\omega)\approx\frac{1}{(2\pi)^{d-1}}\frac{k(\omega)^{d-1}}{U(\omega)}Q(\overline{x,}\omega)\times\int d\Omega_k e^{-ik|\Delta x|\cos\theta} \tag{25}$$

where θ is the angle between Δx and $\overline{k,}$ and $k=k(\overline{x,}\omega)$ is found implicitly from $\omega=\nu(\overline{x,}k)$. The angular integral can be computed in terms of Bessel functions to give

$$\widetilde{C}^{\rm sm}(x_1,x_2,\omega)\approx\frac{\Omega_d}{(2\pi)^{d-1}}\frac{k(\omega)^{d-1}}{U(\omega)}\times\left[\Gamma\left(\frac{d}{2}\right)\frac{J_{d/2-1}(k|x_1-x_2|)}{(k|x_1-x_2|/2)^{d/2-1}}\right]Q(\overline{x,}\omega). \tag{26}$$

This is very similar to a result obtained by Berry [1,11] for spatial correlations in quantum mechanical wave functions when the classical dynamics is chaotic. The spatial dependence is dominated by the term in square brackets, which equals 1 when $x_1=x_2$ and decays in an oscillatary manner to zero as x_1 and x_2 get further apart. When $x_1=x_2$ the result is consistent with Eq. (23). As the points move away from each other, the function oscillates with spatial separation with a period equal to the local wavelength $\lambda(\overline{x,}\omega)$, and with an amplitude that decays at the rate $|\Delta x|^{(d-1)/2}$. Remember that this oscillation is in the spatial variables only — the dependence on ω is smooth when $x_1=x_2$ and only oscillates weakly (through the Bessel function) as the points are moved apart. This completes our discussion of the smooth dependence on ω.

C. The fluctuating contribution

We return now to the calculation of cross terms $\alpha\neq\beta$ in the correlation function $\widetilde{C}(x_1,x_2,\omega)$, where we once again allow x_1 and x_2 to be distinct. The stationary phase approximation of Eq. (13) is complicated by the fact that the stationary phase points are not isolated, but occur in one-dimensional families, much like in the calculation of periodic orbit contributions to the Gutzwiller trace formula [1]. As in that case, the total integral is broken into an integral of $d-1$ coordinates transverse to the orbit, for which stationary phase analysis is possible, followed by an integral along the length of the orbit for which there is no oscillating phase and which must be computed exactly. The result is a sum of the form

$$\widetilde{C}(x_1,x_2,\omega)=\sum_\gamma B_\gamma(x_1,x_2,\omega)e^{iR_\gamma(x_1,x_2,\omega)}, \tag{27}$$

where γ labels the trajectories from x_1 to x_2 (in positive or negative time) and $R_\gamma(x_1,x_2,\omega)$ is the real part of the action. The amplitude $B_\gamma(x_1,x_2,\omega)$ is an integral over the past of the trajectory, involving the amplitudes of the contributing orbits α and β, and a Hessian matrix of the phase. It is displayed in Appendix D. By ''past'' we mean here those parts of the trajectory preceding both x_1 and x_2. Though Eq. (27) above looks formally very much like Eq. (11) for the Green's function, there is one important difference. Causality restricts the orbits contributing to $G(x,x',\omega)$ to those that pass through x' before x, whereas in the approximation above for $\widetilde{C}(x_1,x_2,\omega)$, all orbits between x_1 and x_2 contribute, even those passing through x_1 first. Other than that the structure is similar, though. In fact, it is shown in Appendix D that $B_\gamma(x_1,x_2,\omega)$ is proportional to the amplitude $A_\gamma(x_1,x_2,\omega)$.

The integral one finds for $B_\gamma(x_1,x_2,\omega)$ immediately following the stationary phase integral is a rather complicated one over a single spatial coordinate $x'_\parallel$ labeling points on the past of γ. However, when reparametrized over the time coordinate t' the integrand simplifies significantly. As shown in Appendix D, the result is that the amplitude $B_\gamma(x_1,x_2,\omega)$ reduces to a product of $A_\gamma(x_1,x_2,\omega)$, the amplitude that would appear in the Green's function $G(x_1,x_2,\omega)$ (and independent of t'), and an integral of the source over the past of the trajectory:

$$B_\gamma(x_1,x_2,\omega)=-iA_\gamma(x_1,x_2,\omega)\int_{-\infty}^0 dt'|J'|\times e^{-K_\alpha[x_1,x'(t'),\omega]-K_\beta[x_2,x'(t'),\omega]}\times P_\eta(x'(t'),\omega). \tag{28}$$

Strictly speaking, one might define the Green's-function amplitude $A_\gamma(x_1,x_2,\omega)$ only if x_2 precedes x_1 on the trajectory. If the reverse occurs we can, if necessary, define the amplitude through $A_\gamma(x_1,x_2,\omega)=A^*_{\widetilde{\gamma}}(x_2,x_1,\omega)$, where $\widetilde{\gamma}$ is the reversal of the orbit γ. The integral over t' in Eq. (28) is taken up to the intersection of the trajectory with whichever of the points x_1 or x_2 comes first, and this is where $t'=0$ is defined. Let $Q_\gamma(x_1,x_2,\omega)$ be the function defined in Eq. (19), evaluated for this initial point of the trajectory — the arguments are changed because we label the trajectory with (x_1,x_2,ω) and γ rather than the initial condition in phase space. Then we can write

$$B_\gamma(x_1,x_2,\omega)=-iA_\gamma(x_1,x_2,\omega)e^{-|K_\gamma(x_1,x_2,\omega)|}Q_\gamma(x_1,x_2,\omega). \tag{29}$$

The function $Q_\gamma(x_1,x_2,\omega)$ can be thought of as an accumulated excitation of the trajectory γ over its past by the field η.

Let us now specialize these results to the power spectrum $P_u(x,\omega)$ by setting $x_2=x_1=x$. In this case we get contributions from all trajectories that pass through and return to x. Such orbits occur in time-reversed pairs and their contributions are complex conjugates of each other. Pairing these together in the sum we get

$$P_u^{\rm osc}(x,\omega)=\sum_\gamma{}' \; 2|A_\gamma(x,x,\omega)|e^{-|K_\gamma(x,x,\omega)|}\times\cos\left[R_\gamma(x,x,\omega)-\frac{\mu_\gamma\pi}{2}-\frac{(d-1)\pi}{4}\right]Q_\gamma(x,x,\omega). \tag{30}$$

The prime indicates that this sum is restricted to trajectories in positive time — the time-reversed orbits are included in the argument of the sum along with them. We have also explicitly included the phase of the amplitude $A_\gamma(x_1,x_2,\omega)$ — μ_γ is the Maslov index of the orbit γ, computed as if it would contribute to a Green's function. In this form, $P_u(x,\omega)$ is obviously real, as it should be. This fluctuating function of ω is to be superimposed on the steady background represented by Eq. (20).

As we did for $P_u^{\rm sm}(x,\omega)$, let us estimate the magnitude of $P_u^{\rm osc}(x,\omega)$ for the special case of billiard problems. Assuming that there is no extra damping of waves from reflections at boundaries, the estimate in Eq. (22) should still be valid for Q_γ. The size of A_γ remains to be estimated. The explicit expression in Eq. (D2) contains a factor that scales like an inverse rescaled group velocity, $J/\dot{x}_\parallel \sim J/[\partial\omega/\partial k] \sim U^{-1}$. There remains the determinant of a matrix that scales as $[k/L]^{(d-1)/2}$, where L is a characteristic length of the system. In two-dimensional systems, for example, L is the distance to the nearest focal point. The amplitude therefore scales according to

$$|A_\gamma| \sim U^{-1}[k/L]^{(d-1)/2}. \tag{31}$$

We find, then, that the magnitude of a typical contribution to $P_u^{\rm osc}$ depends on ω as follows:

$$|B_\gamma| \sim \frac{e^{-\alpha l_\gamma}}{(kL)^{(d-1)/2}} \frac{k^{d-1}}{\alpha U^2} P_\eta(\omega). \tag{32}$$

Here, l_γ is the geometrical length of the orbit. Comparing with Eq. (23), we see that oscillatory contributions are smaller than the smooth part by a factor of order $e^{-\alpha(\omega) l_\gamma}/[k(\omega)L]^{(d-1)/2}$. Notice also that, unlike the smooth part, the oscillating part of the power spectrum (or the correlation function) necessarily has an exponential dependence on the frequency even when measurements are made in the excitation region — this exponential dependence will dominate the oscillating contribution when the decay length α^{-1} is smaller than typical orbit lengths.

V. TRACES

The approximation to the power spectrum takes on a particularly elegant form if we average over position. As with the trace of the Green's function in the Gutzwiller trace formula [1], the result is a sum over periodic orbits, each with a contribution that is canonically invariant and easy to compute. Let us define

$$\overline{P_u}(\omega) = \int dx\ P_u(x,\omega). \tag{33}$$

We will tacitly assume that the dynamics is confined to a bounded region in space, so that this integral makes sense. If the system were unbounded, one could integrate over some subregion and confine the contributions of classical orbits discussed below to those parts that lie in that subregion. If the source power spectrum is homogeneous as in Eq. (10), then $\overline{P_u}(\omega)$ can be expressed in the following invariant way:

$$\overline{P_u}(\omega) = P_\eta(\omega) \mathrm{Tr}[\hat{G}(\omega)\hat{G}^\dagger(\omega)], \tag{34}$$

which justifies the term trace formula for what follows. This is to be compared with the Gutzwiller trace formula, in which the trace of $\hat{G}(\omega)$ rather than of $\hat{G}(\omega)\hat{G}^\dagger(\omega)$ is computed. If the source power spectrum is not homogeneous, then invariance is lost because there are preferred coordinates in which to calculate its contribution, but the broad structure is still that of a trace formula.

We first calculate the smooth part. Integrating over x, Eq. (20) for $P_u^{\rm sm}(\omega)$ becomes

$$\overline{P_u^{\rm sm}}(\omega) = \frac{1}{(2\pi)^{d-1}} \int dx\ dk\, \delta(D(x,k,\omega)) Q(x,k,\omega). \tag{35}$$

This has the same form as the Thomas-Fermi contribution to the density of states [1] — an integral over the ''energy shell'' $\nu(x,k)=\omega$ in phase space with the canonical measure $\int dx dk\, \delta(D(x,k,\omega))$. The only difference is that, here, there is the integrand $Q(x,k,\omega)$.

Calculation of the trace of the oscillating part by the stationary phase condition is the same as the calculation of the Gutzwiller trace formula — the stationary phase condition restricts contributing orbits to those that are periodic, and stationary phase calculation around each periodic orbit then leads to a time integral around the corresponding primitive orbit. There are three differences: orbits with negative time contribute; there is an extra factor of $-iQe^{-|K|}$ for each orbit; there is a factor $|J|$ in the amplitude A that converts any time integral around the orbit from t to τ. The result is the following sum over periodic orbits:

$$\overline{P_u^{\rm osc}}(\omega) = \sum_{\rm po} -i\langle Q\rangle_{\rm ppo} e^{-K_{\rm po}} \times \left\{ i \frac{T_{\rm ppo}}{|\det(M_{\rm po}-I)|^{1/2}} e^{iR_{\rm po} - i\mu_{\rm po}\pi/2} \right\}. \tag{36}$$

The term in curly brackets is what one gets in a straight trace of $\hat{G}(\omega)$ [remember that there is a minus sign in the definition of $\hat{G}(\omega)$ relative to the definition of the Green's function in quantum mechanics, so the factor in front is i, not $1/i$]. $M_{\rm po}$ is the linearized surface of section mapping around the orbit and $\mu_{\rm po}$ is the Maslov index of the orbit as a periodic orbit, not the index μ_γ of the Green's function. The sum over orbits includes reversals of orbits as well as forward iterations. The term $T_{\rm ppo}$ is the integral of $d\tau$, not dt, over the primitive orbit,

$$T_{\rm ppo} = \oint_{\rm ppo} dt |J| = \oint_{\rm ppo} d\tau. \tag{37}$$

The time integral actually occurs with an argument $Q(x(t),k(t),\omega)$, but we have chosen not to put this in the curly brackets to clarify the connection with the Gutzwiller trace formula. To compensate, there is a factor $-i\langle Q\rangle_{\rm ppo} e^{-|K_{\rm po}|}$ in front, where

$$\langle Q\rangle_{\rm ppo} = \frac{1}{T_{\rm ppo}} \oint_{\rm ppo} d\tau Q(x(t),k(t),\omega). \tag{38}$$

We put a subscript ppo on $\langle Q\rangle_{\rm ppo}$ because it does not depend on how many times the periodic orbit is repeated.

Let us reorganize the sum as one first over the primitive periodic orbits, followed by a sum over the number of repetitions r. We couple together positive and negative repetition numbers, as in Eq. (30), to get

$$\overline{P_u^{\rm osc}}(\omega)=2\sum_{\rm ppo}\langle Q\rangle_{\rm ppo}T_{\rm ppo}\sum_{r=1}^{\infty}\frac{e^{-r|K_{\rm ppo}|}}{|\det(M_{\rm ppo}^r-I)|^{1/2}}\times\cos\left[r\left(R_{\rm ppo}-\frac{\mu_{\rm ppo}\pi}{2}\right)\right]. \tag{39}$$

Everything here is canonically invariant, except for $\langle Q\rangle_{\rm ppo}$, which singles out configuration space coordinates if the source is not homogeneous.

We conclude this section with estimates for the relative magnitudes of $\overline{P_u^{\rm sm}}(\omega)$ and $\overline{P_u^{\rm osc}}(\omega)$ in the special case of billiard systems. Equation (22) still holds for $\overline{P_u^{\rm sm}}(\omega)$, except now there is a factor of V, the d-dimensional volume of the system in configuration space. As for $\overline{P_u^{\rm osc}}(\omega)$, Eq. (23) works for $\langle Q\rangle_{\rm ppo}$, and $T_{\rm ppo}\sim l_{\rm ppo}/U$. Putting this together, and comparing, we see that the amplitude of a typical contribution to $\overline{P_u^{\rm osc}}(\omega)$ is of the order

$$\overline{P_u^{\rm osc}}(\omega)\sim\frac{kl_{\rm ppo}}{k^dV}e^{-\alpha l_{\rm ppo}}\overline{P_u^{\rm sm}}(\omega). \tag{40}$$

For orbits of similar length, or in regimes where damping over the length of an orbit is insignificant, this relative size is smaller than the ratio $e^{-\alpha l_\gamma}/(kL)^{(d-1)/2}$ that we calculated for $P_u(x,\omega)$. Therefore, we see that integrating over x has the effect of decreasing the importance of the oscillations relative to the smooth background.

VI. APPLICATION TO THE HYDRAULIC JUMP

Let us show how the theory developed here can explain the power spectrum in Fig. 1. A thorough analysis is given in Ref. [7], but here we will confine ourselves to the main features, to illustrate that the formulas work.

Linear propagation of surface waves in water is a standard problem of hydrodynamics [12], treated by assuming that the velocity field is derived from a potential $\mathbf{u}=\nabla\phi$ for which the continuity equation reduces to Laplace's equation, and then imposing time-dependent boundary conditions on ϕ by balancing forces at the surface. The equations are messy, but yield a simple dispersion relation [12]:

$$\omega^2=\left(g+\frac{Tk^2}{\rho}\right)k\tanh kh, \tag{41}$$

where h is the depth of the water, T the surface tension, ρ the density, and g the acceleration due to gravity. In the hydraulic jump experiment we should in principle also include a Doppler shift arising from the radial flow of the water, but this flow velocity is always smaller than the group and phase velocities of the waves and therefore has a small effect, so we ignore it.

This dispersion relation is enough to determine most of the dynamics governing the geometrical limit of the wave equation. As long as the depth of the water is uniform, or deep enough that $\tanh kh\approx 1$, the dispersion relation is independent of position and the geometrical rays are straight lines. This is essentially the case in the experiment — variation in $\tanh kh$ is only significant below about 20 Hz in the power spectrum shown, and even then the depth of the water is uniform to within about 20% outside the jump. Therefore it is a good working assumption that the rays are in the form of straight lines in the interior of the system. It remains to determine what happens to rays at the boundaries, which, in the case of Fig. 1, are (a) an inner circle formed by the jump itself, and (b) an outer boundary imposed by placing a barrier. A complete treatment of wave propagation at either of these boundaries is forbiddingly difficult — even determining the shape of the unperturbed flow before adding surface oscillations is a serious problem. We therefore adopt a simplified treatment. We treat the outer boundary as sharp, reflecting rays specularly with some unknown phase shift — empirically one can see that waves are reflected with little loss of amplitude. The jump itself is more problematic. When the spectrum is analyzed, no evidence is found for the contribution of rays that are incident on the jump. Presumably, all such rays are so long that the dissipation factor damps them out completely. It may also be that there is enhanced damping at reflection from the jump contributing to their nonappearance. In practical terms, we do not need a detailed explanation of these reflections, and we do not offer one.

In a complete analysis we would also want to know the equivalent of the dispersion function $D(x,k,\omega)$ to determine how the wave couples to the source. We do not do this because in our case the source does not have any intrinsic meaning of its own. The model of a source forcing the wave equation was used as a simplistic replacement for the very complex problem of the radial flow outside the jump coupling with an irregular and nonlinear flow in the jump itself. In a sense the source represents all the missing, but important, terms that were excluded from the hydrodynamic equations near the jump. We will satisfy ourselves with the analysis that can be achieved with the dispersion relation alone.

The smooth part of the power spectrum is obtained by integrating over all orbits that go directly from the jump to the point of measurement. The most significant aspect of this contribution is that it will be damped at a rate $e^{-2\alpha(\omega)l}$ where $\alpha(\omega)$ is the inverse decay length due to viscous damping [7] (not included in the dispersion relation above) and l is the distance from the jump to the point of measurement. This decay dominates the ω dependence of the power spectrum at high frequencies. For the geometry considered, it is found that there is only one orbit contributing significantly to the oscillating part. This is the radial orbit that goes to the outer boundary and is reflected straight back. The contribution of this orbit is of the form

$$P_u^{\rm osc}(\omega)=F(\omega)e^{-\alpha(\omega)[2l+L]}\cos[k(\omega)L+\theta], \tag{42}$$

where $L/2$ is the distance from x to the outer edge, $F(\omega)$ is an unknown amplitude, and θ an unknown phase. We have separated out the exponential decay from the Q factor

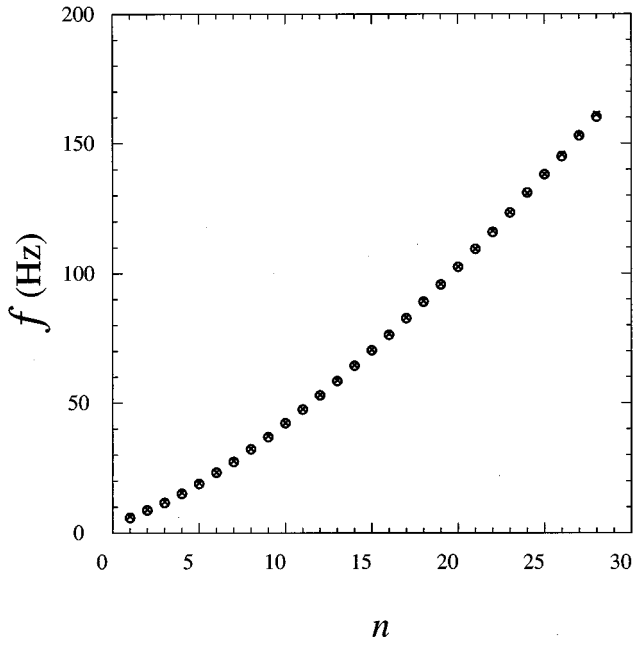

FIG. 3. The frequencies at which maxima appear in the power spectrum in Fig. 1 are plotted as circles. Crosses show the peaks of Eq. (42) predicted by the dispersion relation in Eq. (41). The length used was $L=8.0$ cm, which is consistent with the geometry of the experiment. The height effects only the lowest peaks and the average experimental value, $h=1.5$ mm, was used. The phase $\theta=176°$ was chosen to fit the data.

$(e^{-2\alpha l})$ and the decay over the orbit itself $(e^{-\alpha L})$. Having done this, the remaining amplitude $F(\omega)$ should depend more weakly on ω.

The simplest check of the oscillating contribution is to examine the spacing $\Delta\omega$ between peaks. It is given by

$$\Delta\omega=\frac{2\pi}{L}\frac{\partial\omega}{\partial k}. \tag{43}$$

We immediately see now why the spacing between peaks increases as the point of measurement approaches the edge $(L\to 0)$. The gradual increase of the spacing with frequency is also simple to explain. As long as the wavelength is less than about 2 cm (and this is usually the case), the capillary term dominates over gravity in the dispersion relation and $\omega^2\sim k^3$, so $\Delta\omega\sim\omega^{1/3}$, as is observed empirically. For a more detailed comparison, we compare in Fig. 3 the positions of the peaks in the power spectrum with the prediction of Eq. (42). For reasonable choices of L and θ, quantitative agreement is obtained over a wide range of frequencies.

VII. CONCLUSION

Following observation in a hydraulic jump experiment, we have shown that interference effects arise in a natural way in the power spectrum of any variable that evolves according to a wave equation. The interference effects are evident when the wave equation is driven by a source that itself has a power spectrum that varies smoothly over a large range of frequencies. In this case the power spectrum of the wave variable consists of a smooth background, similar to the source power spectrum, upon which are superimposed fluctuations in frequency that can be related to trajectories obtained from the geometrical limit of the wave equation.

The connection between the source and wave power spectra was made under the assumption that spatial correlations in the source have length scales much shorter than a wavelength. Using a semiclassical approximation of the Green's function, we arrived at straightforward expressions for the smooth and fluctuating parts of the wave power spectrum. The smooth part is an integral over all trajectories of a given frequency passing through the point in space where the measurement is taken. Each trajectory donates a contribution Q given by an integral over its past of the source power, modulated by an exponential decay arising from dissipation. The fluctuating part is a discrete sum over all trajectories that begin and end at the point of measurement. Each contributes a term with a phase and amplitude closely related to those of the Green's function, but with an additional factor Q similar to the term arising in the smooth part.

We also derived trace formulas that apply to an average of the wave power spectrum over position. These formulas are canonically invariant, and are very similar to the Gutzwiller trace formula for the density of states in quantum mechanics. The smooth part is an integral in canonical coordinates over an ''energy shell'' in phase space, formed by fixing the frequency, with an excitation factor Q in the integrand. The fluctuating part is a sum over all periodic orbits at the given value of frequency. The contribution of each is the same as the contribution that would occur in the Gutzwiller trace formula, except there is once again a factor of Q, this time averaged over the periodic orbit.

This picture enabled us to explain the main features of the power spectra that had been obtained in the hydraulic jump experiment. There is one important orbit there, going radially from the point of measurement out to the edge and back. This produced a single sinusoidal oscillation in the power spectrum, whose local period agreed well with the result predicted from the dispersion relation. We suspect that oscillations like this may appear in the power spectra of other wave systems. Examples are fluctuations in the electromagnetic field of a blackbody and in acoustic emission signals measured in finite systems.

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APPENDIX A: EXPLICIT SEMICLASSICAL GREEN'S FUNCTION

In this appendix we will provide a detailed discussion of the semiclassical approximation for the Green's function $G(x,x',\omega)$ appearing in Eq. (11), including explicit formulas for the amplitudes $A_\alpha(x,x',\omega)$ *etc.* In order to write down a complete approximation for the most general wave operator $\hat{\mathcal{L}}(\omega)$, the most efficient approach is to reformulate the problem so that it is directly equivalent to the definition of Green's functions in quantum mechanics. Then the standard

formulas derived in that context can be quoted directly.

To formulate the problem in this way, we introduce an extra parameter λ, playing the role of energy, and define a Green's function $\widetilde{G}(x,x',\omega,\lambda)$ by

$$\{\lambda-\mathcal{L}(i\omega,\nabla,x)\}\widetilde{G}(x,x',\omega,\lambda)=\delta(x-x'). \tag{A1}$$

This gives us the Green's function we want when $\lambda=0$:

$$G(x,x',\omega)=-\widetilde{G}(x,x',\omega,0). \tag{A2}$$

Though evaluated at $\lambda=0$, the semiclassical approximations will involve derivatives in which λ varies infinitesimally. The parameter λ controls the manner in which the frequency ω appears in the approximation.

Insertion of an eikonal ansatz into the λ-dependent equation yields a Hamilton-Jacobi equation of the form

$$\lambda-D(x,k,\omega)=0, \tag{A3}$$

where the dispersion function $D(x,k,\omega)$ plays the role of a phase-space Hamiltonian, corresponding to the operator Hamiltonian $\hat{\mathcal{L}}(\omega)$. In this equation ω appears as a parameter and does not play a dynamical role. We can invert this equation to write $\omega=\nu(x,k,\lambda)$ [in this section we find it convenient to distinguish between phase space functions, $D(x,k,\omega)$ and $\nu(x,k,\lambda)$, and the values they take on, λ and ω—the distinction may be relaxed elsewhere]. It turns out that either of the two functions, $D(x,k,\omega)$ or $\nu(x,k,\lambda)$ gives the same paths in phase space when used as Hamiltonians. However, the time increments, $d\tau$ and dt respectively, parametrizing the paths are different. They are related to each other by

$$d\tau=J(x,k)dt, \tag{A4}$$

where the Jacobian $J(x,k)$ can be calculated from one of two equivalent forms:

$$J(x,k)=-\left.\frac{\partial\nu(x,k,\lambda)}{\partial\lambda}\right|_{\lambda=0}=-\left(\left.\frac{\partial D(x,k,\omega)}{\partial\omega}\right|_{\omega=\nu(x,k)}\right)^{-1}. \tag{A5}$$

This is verified by writing out Hamilton's equations and implicitly differentiating through Eq. (A3). We are therefore given a choice between describing the dynamics in terms of the variables (λ,τ) or (ω,t). While the results are initially given to us in terms of (λ,τ) and indeed are most compactly written in terms of those variables, the variables (ω,t) are more physical. For example, one can often write a local dispersion relation $\omega=\nu(x,k)$ (with $\lambda=0$) more easily than writing the wave equation in a form that has an obvious semiclassical limit, as with water waves. We will therefore endeavor to quote the final results in terms of (ω,t), despite the notational inconvenience.

Now we are ready to discuss the semiclassical approximation to $\widetilde{G}(x,x',\omega,\lambda)$. It has the same form as Eq. (11), a sum over trajectories α going from x' to x at fixed values of λ and ω. The amplitude of each contribution α is of the form

$$\widetilde{A}_\alpha(x,x',\omega,\lambda)=\frac{1}{i}\frac{1}{(2\pi i)^{(d-1)/2}}\left|\left(\frac{\partial(k',\tau)}{\partial(x,\lambda)}\right)_{(x',\omega)}\right|^{1/2}e^{-i\mu\pi/2}, \tag{A6}$$

where (x',k') are the coordinates of the initial point and (x,k) those of the final point. We denote the time of the orbit by τ in D dynamics and by t in ν dynamics. The subscripts (x',ω) on the Jacobian indicate that those variables are held fixed while the derivatives are taken. A detailed discussion of the semiclassical Green's function can be found in Ref. [1].

We want to calculate the amplitude in $G(x,x',\omega)$, given by $A_\alpha(x,x',\omega)=-\widetilde{A}_\alpha(x,x',\omega,0)$, in such a way that λ does not appear explicitly. This is achieved by certain manipulations of the Jacobian in the amplitude of Eq. (A6), which we do not show in detail. We just quote the result:

$$A_\alpha(x,x',\omega)=\frac{i}{(2\pi i)^{(d-1)/2}}|JJ'|^{1/2}\left|\left(\frac{\partial(k',t)}{\partial(x,\omega)}\right)_{x'}\right|^{1/2}e^{-i\mu\pi/2}. \tag{A7}$$

The dispersion relation $\omega=\nu(x,k)$ is enough to determine everything in this amplitude except the factors J' and J, which are obtained by evaluating Eq. (A5) at the initial and final points of the trajectory, respectively. The main part of the amplitude should remain unchanged even if we consider very different types of wave equations such as integral equations or equations with free boundary conditions. With the assumption that the wave equation was a PDE we arrived at a definite expression for the remaining Jacobians, which we interpret as telling us how the source couples locally to the wave field. For other types of wave equations it just remains to calculate the analog of this coupling factor, and the rest of the calculations can then proceed in direct analogy with those discussed in the main text, presumably without qualitative differences.

APPENDIX B: IMAGINARY PART OF THE ACTION

The imaginary part K of the action of an orbit is easily treated at the level of a first order perturbation about a real orbit. The real orbit is first computed using only the real part of the dispersion relation. The perturbation due to the imaginary part is then calculated as a special case of the following general relationship for the variation of an action with a system parameter:

$$\frac{\partial S(E;,\alpha)}{\partial\alpha}=-\int_{\text{orbit}}dt\frac{\partial H}{\partial\alpha}. \tag{B1}$$

Here H is any Hamiltonian depending on a parameter α, and $S(E;\alpha)$ is the action of an orbit with energy E that is periodic or has end points fixed in configuration space. An explicit proof of this is given, for example, in [13]. Applying this to the complex dispersion relation, we obtain the following approximate expression for K,

$$K\approx-\int_{\substack{\text{real}\\ \text{orbit}}}dt\ \mathrm{Im}(\nu), \tag{B2}$$

where the integral is taken over the real orbit obtained by using $\mathrm{Re}[\nu(x,k)]$ as a Hamiltonian.

APPENDIX C: EXPLICIT CALCULATION OF $P_u^{\mathrm{sm}}(x_0,\omega_0)$

In this Appendix we fill the gaps between Eqs. (15) and (16), expressing the smooth part of the approximation to

$P_u^{\rm sm}(x,\omega)$ in terms of more natural variables than arise in the initial substitution of the semiclassical approximation.

The first step is to expand the space over which integration takes place to include frequency and final position. Let (x_0,ω_0) be the particular values of the variables (x,ω) at which we want to evaluate the power spectrum. Then we can rearrange the measure appearing in Eq. (16) as follows:

$$\sum_\alpha \int dx' |A_\alpha(x_0,x',\omega_0)|^2 = \sum_\alpha \int dx dx' d\omega\, \delta(x-x_0)\,\delta(\omega-\omega_0)|A_\alpha(x,x',\omega)|^2. \tag{C1}$$

The variables (x,x',ω), along with the index α, can be regarded as coordinates for the $(2d+1)$-dimensional space of all possible trajectories. More natural variables for the space of trajectories would be (x',k',t), the coordinates of the initial point and the time of the orbit. The amplitude term $|A_\alpha(x,x',\omega)|^2$ above contains the Jacobian for precisely this coordinate transformation, turning $\Sigma_\alpha \int dx dx' d\omega$ into $\int dx' dk' dt$. Taking Eq. (A7) and putting the variable x' into the Jacobian proper we get

$$|A_\alpha(x,x',\omega)|^2 = \frac{1}{(2\pi)^{d-1}}|JJ'|\left|\left(\frac{\partial(x',k',t)}{\partial(x',x,\omega)}\right)_{x'}\right|, \tag{C2}$$

which allows us to take Eq. (C1) one step further:

$$\sum_\alpha \int dx' |A_\alpha(x_0,x',\omega_0)|^2 = \frac{1}{(2\pi)^{d-1}}\int dx' dk' dt\, \delta(x-x_0)\,\delta(\omega-\omega_0)|JJ'|. \tag{C3}$$

As mentioned in Appendix A, the presence of the Jacobians $|JJ'|$ is a penalty to be paid for using the physical variables (ω,t) rather than the mathematically more convenient (λ,τ). Had we written the result in terms of the measure $\int dx' dk' d\tau \delta(x-x_0)\,\delta(\lambda)$, they would not be there.

This proves Eq. (16).

APPENDIX D: EXPLICIT CALCULATION OF $\tilde{C}(x_1,x_2,\omega)$

In this Appendix we supply the calculation leading to the expression for $B_\gamma(x_1,x_2,\omega)$ in Eq. (28). Let $x'_\parallel$ and $x'_\perp$ be the coordinates perpendicular and parallel, respectively, to the trajectory at x'. Computation of the integral in Eq. (13) along the $x'_\perp$ coordinates by the stationary phase approximation gives

$$B_\gamma(x_1,x_2,\omega) = \left(\frac{2\pi}{i}\right)^{(d-1)/2} e^{iN_+\pi/2}\int dx'_\parallel A_\alpha(x_1,x',\omega) \times A^*_\beta(x_2,x',\omega)\left|\frac{\partial^2(R_\alpha-R_\beta)}{\partial x'_\perp \partial x'_\perp}\right|^{-1/2} \times e^{-K_\alpha(x_1,x',\omega)-K_\beta(x_2,x',\omega)} P_\eta(x',\omega), \tag{D1}$$

where N_+ is the number of positive eigenvalues of the symmetric matrix $\partial^2(R_\alpha-R_\beta)/\partial x'_\perp \partial x'_\perp$. The next step is to change the integration variable from $x'_\parallel$ to a more natural time variable t'.

The calculations that follow can be performed by explicit manipulation of the Jacobian and Hessian matrices as they appear in Eq. (D1). However, much of the detail can be sidestepped if we appeal instead to an analogy with a very similar expression for the composition of propagators in quantum mechanics, which we do in this appendix.

When the spatial arguments of the Green's function are restricted to (possibly distinct) surfaces of section in phase space, the amplitudes and phases in the sum over orbits take a form almost exactly the same as those of a quantum propagator between two different times t and t'. The amplitude for contribution to the Green's function of an orbit α going from x' in the surface of section Σ' to x in Σ can be factored as follows:

$$A_\alpha(x,x',\omega) = i\left|\frac{JJ'}{\dot{x}_\parallel \dot{x}'_\parallel}\right|^{1/2} \times\left\{\frac{1}{(2\pi i)^{(d-1)/2}}\left|\left(\frac{\partial k'_\perp}{\partial x_\perp}\right)_{x',x_\perp,\omega}\right|^{1/2} e^{-i\mu_\alpha\pi/2}\right\}. \tag{D2}$$

The term in curly brackets is formally identical to the amplitude of the contribution of an orbit to a quantum mechanical propagator $K(x,x',t,t')=\langle x|U(t,t')|x'\rangle$ in the Van Vleck approximation. Here $U(t,t')$ is the unitary time evolution operator from time t' to time t. This analogy has been useful in quantum theory [14] and will allow us here to borrow standard results for the Van Vleck formula to write an immediate expression for $B_\gamma(x_1,x_2,\omega)$.

The result we want to borrow is the statement that the rule for the concatenation of time evolution operators

$$U(t_2,t_1)=U(t_2,t')U(t_1,t')^\dagger \tag{D3}$$

carries over to the Van Vleck approximation if we use the stationary phase approximation to compound the operators, giving

$$K(x_2,x_1,t_2,t_1)\approx \int dx' K(x_2,x',t_2,t')K(x_1,x',t_1,t')^*. \tag{D4}$$

The trace over x' here is formally identical to the trace over $x'_\perp$ in the calculation leading to Eq. (D1). Except for the additional terms outside curly brackets in Eq. (D2), the manipulation of amplitudes and phases that lead to Eq. (D4) applies equally well to Eq. (D1) and we can write

$$\left(\frac{2\pi}{i}\right)^{(d-1)/2} e^{iN_+\pi/2} A_\alpha A^*_\beta \left|\frac{\partial^2(R_\alpha-R_\beta)}{\partial x'_\perp \partial x'_\perp}\right|^{-1/2} = -i\left|\frac{J'}{\dot{x}_\parallel'}\right| A_\gamma(x_1,x_2,\omega). \tag{D5}$$

Once we recognize the Jacobian for the variable change $x'_\parallel \to t'$, Eq. (28) follows immediately.

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